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LETTER TO THE EDITOR

# Impact of localization on Dyson's circular ensemble

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**Abstract.** A wide variety of complex physical systems described by unitary matrices have been shown numerically to satisfy level statistics predicted by Dyson's circular ensemble. We argue that the impact of localization in such systems is to impose certain restrictions on the eigenvalues. We consider a solvable model which takes into account such restrictions qualitatively and find that within the model a gap is created in the spectrum, and there is a transition from the universal Wigner distribution towards a Poisson distribution with increasing localization.

A characteristic statistical property of chaotic (as opposed to integrable) states in quantum systems is the distribution of their energies. In particular, the nearest-neighbour spacing distribution or the long-range spectral rigidity of a local set of levels for a wide variety of systems in the chaotic regime agree remarkably well with the universal Wigner distributions obtained from the Gaussian random matrix theory [1, 2]. The same is also true for ergodic quasi-energy eigenstates for a variety of periodically driven systems [3] described by the Flouquet matrix, whose eigenvalues lie on a complex unit circle, and belong to Dyson's 'circular' ensemble [4]. We will reserve the term Wigner ensemble for eigenvalues on the real line. Both ensembles follow the same Wigner distributions in the limit of large number of eigenvalues.

A new problem in this area is the impact of localization on the statistical properties of chaotic eigenstates, which leads to deviations from the universal Wigner distributions. Whilst attempts have been made to generalize the Wigner ensemble to include such deviations at a phenomenological level by imposing suitable constraints [5, 6], it is clear that such constraints cannot affect the circular ensemble in the same way because the eigenvalues are already bounded. Nevertheless, numerical studies involving the scattering matrix for disordered conductors [7], as well as the Flouquet matrix for periodically driven systems [8], show similar deviations in the spectral properties [9]. It is therefore worthwhile to consider an analytic model that can accomodate such deviations in the circular ensemble.

In this letter, by considering the scattering matrix describing a disordered conductor as an example, we will argue that the qualitative effect of localization on the statistical properties of the circular ensemble is to impose certain restrictions on the eigenvalues. We will then construct a solvable model that takes into account these restrictions in a qualitative way, and show that this leads to a transition in the spectral properties from the universal Wigner distribution towards a Poisson distribution as a function of a single parameter related to localization.

Let us consider a one-dimensional scattering of plane waves of energy $E$ from a potential barrier of width $a$ and height $V_0$. Define $\hbar k_0 = \sqrt{2mE}$ and $\hbar k = \sqrt{2m(E - V_0)}$, where $m$ is the mass of the incident particle. The $2 \times 2$ scattering matrix $S$ has the simple form

$$S = e^{-i\psi} \begin{pmatrix} \cos\theta & -i\sin\theta e^{-ik_0 a} \\ -i\sin\theta e^{+ik_0 a} & \cos\theta \end{pmatrix}$$

where

$$\cos\theta = 2k_0/\sqrt{4k_0^2 + k^2\sin^2(ka)[1 - k_0^2/k^2]} \qquad \psi = k_0 a + \mu \qquad \cos\mu = \cos\theta\,\cos ka.$$

The eigenvalues are $e^{-i\psi \pm i\theta}$. In a very crude way, we might think of the case $E > V_0$ to mimic a metal, with plane-wave states in the region $0 < x < a$, while the case $E < V_0$ will mimic a finite length insulator with exponentially localized states in the region. It is clear that while in the former case the quantity $\cos\theta$ can take on all values from zero to unity as $k_0$ is varied, it becomes restricted to values less than unity in the latter case, where $k = ip$ is imaginary and the term $k^2\sin^2(ka)[1 - k_0^2/k^2]$ is replaced by $p^2\sinh^2(pa)[1 + k_0^2/p^2]$. Such a restriction can be interpreted as a constraint on the possible maximum of $\mathrm{Tr}(S + S^\dagger)$ which is proportional to $\cos\theta$, and the restriction increases with increasing 'localization' of the waves inside the barrier. In case of a many-channel quasi-one-dimensional conductor, we can think of the various channels as having different incoming energies, and an ensemble of conductors corresponding to different possibilities for the values of $k$. Channels in the metallic regime will correspond to having all possible values of $\theta$ and therefore the eigenvalues will be uniformly distributed on the complex unit circle without any restriction. On the other hand, if the channels are localized, the eigenvalues will be distributed in a way consistent with the restriction on the trace as mentioned above. This very crude argument suggests that at a phenomenological level, the impact of localization on the eigenvalue distribution of scattering matrices can be incorporated by imposing constraints on $\mathrm{Tr}(S + S^\dagger)$. This can be done in a way suggested by Balian [10], namely by introducing Lagrange multiplier functions as constraints in the joint probability distribution of eigenvalues. In the present work we will choose a constraint that has the qualitative features described above, and for which one can, at least in principle, solve for all $n$-point correlation functions of the eigenvalue distribution. The hope is that the qualitative effects obtained from such a solvable model will be independent of the particular choice of the model. Indeed we will show that the model predicts a transition from the highly-correlated Wigner distribution towards an uncorrelated Poisson distribution in a way that is qualitatively similar to the transition seen numerically for a variety of systems.

For eigenvalues on the complex unit circle, Dyson's circular ensemble is based on the basic ansatz of the random-matrix theory that for a physical system described by an $N \times N$ matrix $S$ with eigenvalues $e^{i\theta_n}, n = 1, \ldots, N$, the joint probabilty distribution for the ensemble of all random $S$ matrices consistent with given symmetries (unitarity, time reversal, etc) can be written quite generally in the form [1]

$$P(\theta_1, \ldots, \theta_N) = \prod_{m<n} \left|e^{i\theta_m} - e^{i\theta_n}\right|^\alpha \prod_m w(\theta_m). \tag{1}$$

Here $\alpha$ is a symmetry parameter and is equal to 1, 2 or 4 for orthogonal, unitary and symplectic ensembles, respectively. The function $w(\theta)$ is a Lagrange multiplier function which might take care of any system-dependent physical constraint [10], and in general may depend on various physical parameters. Note that for unbounded eigenvalues of the Wigner ensemble such a constraint is required to keep the distribution normalizable. For the circular ensemble the above distribution is already normalizable for $w(\theta) =$ constant and there is in general no need for additional constraint terms. Dyson has shown explicitly that the two-level correlation function for the above distribution for $w(\theta) = 1/2\pi$ is identical to that of the Wigner ensemble for unbounded eigenvalues in the large-$N$ limit, and therefore leads to the same universal Wigner distributions. However, this distribution is valid only in the weakly disordered or chaotic regime, and as we argued before, the impact of localization can be accommodated phenomenologically by choosing a Lagrange multiplier function constraining $\mathrm{Tr}(S + S^{\dagger})$, or equivalently $\cos\theta$. Because we have no microscopic model at this point, we will choose the constraint, with the correct qualitative features, such that the model is exactly solvable.

Our model corresponds to the choice

$$w(\theta) \sim (1 - \cos\theta)^{N/\lambda}. \tag{2}$$

Clearly this has the qualitative features mentioned above, where the parameter $\lambda$ will serve as a measure of localization; decreasing $\lambda$ increases the constraint on $\cos\theta$. We will show that this model is solvable in the sense that the spectral correlations can be written down in terms of known functions. It turns out that a more general model with two independent parameters, which contains our model (2) as a special limiting case, is also exactly solvable. It is because of its simplicity as well as its possible relation to other problems in physics, that we will start with the more general model, write down the general solution, and will come back to our special limiting case when we analyse and interpret the solution.

The more general two-parameter model is defined by the choice

$$w(\theta) = \frac{1}{2\pi} \left| \frac{(q^{1/2}\mathrm{e}^{\mathrm{i}\theta}; q)_\infty}{(aq^{1/2}\mathrm{e}^{\mathrm{i}\theta}; q)_\infty} \right|^2 \qquad 0 < q < 1 \quad a^2 q < 1 \tag{3}$$

where we have used the notation $(x; q)_n = \prod_{k=0}^{n}(1 - xq^k)$. With the choice $a = q^{N/\lambda}, \lambda \gg 1$, and $q = \mathrm{e}^{-1/N}$ in the limit $N \to \infty$, or equivalently $q \to 1$, we obtain $w(\theta) = (1/2\pi)2^{N/\lambda}(1 - \cos\theta)^{N/\lambda}$ [11], which is our model defined in (2). We will first obtain the general solution for the model (3), and show that only in the above special limit the impact of localization becomes observable in the spectral correlations. In particular, we will show that in this limit a gap appears in the density. We will also show as an explicit example that in this case the number variance obtained from the two-level function shows deviations from the Wigner distribution, towards a Poisson limit. Note that in the other limit $a = 0$ and $q \to 0$, $w(\theta) \to 1/2\pi$, and the model reduces to Dyson's circular ensemble.

For simplicity, we will consider only the case where the symmetry parameter $\alpha = 2$, corresponding to the case without time reversal symmetry. We use the method of orthogonal polynomials [1] and write the product term $\prod_{m<n} |\mathrm{e}^{\mathrm{i}\theta_m} - \mathrm{e}^{\mathrm{i}\theta_n}|$ as a Vandermonde determinant whose elements form a set of polynomials orthogonal with respect to the measure $w(\theta)$. For our particular choice of $w(\theta)$ given in (3), these are the (normalized) Szegö polynomials generalized by Askey [11]:

$$\begin{aligned} \Phi_n(\mathrm{e}^{\mathrm{i}\theta}; q) &= q^{n/2} \left[ \frac{(q,q)_n (q,q)_\infty (a^2q,q)_\infty}{(a^2q,q)_n (aq,q)_\infty (aq,q)_\infty} \right]^{1/2} S^a{}_n \\ S^a{}_n &= \sum_{k=0}^{n} \frac{(aq;q)_k (a,q)_{n-k} (q^{-1/2}\mathrm{e}^{\mathrm{i}\theta})^k}{(q;q)_k (q;q)_{n-k}}. \end{aligned} \tag{4}$$

The polynomials satisfy the orthogonality relation

$$\frac{1}{2\pi} \int_0^{2\pi} \Phi_m(\mathrm{e}^{\mathrm{i}\theta}; q) \overline{\Phi_n(\mathrm{e}^{\mathrm{i}\theta}; q)} w(\theta)\, \mathrm{d}\theta = \delta_{m,n} \tag{5}$$

where the overline denotes complex conjugate. In terms of these polynomials the two-level correlation function is given by [1]

$$K_N(\theta,\phi) = \sqrt{w(\theta)}\sqrt{w(\phi)} \sum_{k=0}^{N-1} \overline{\Phi_k(e^{i\theta})}\Phi_k(e^{i\phi}). \tag{6}$$

We now use the unit-circle analogue of the Christoffel–Darboux identity [16]

$$\sum_{k=0}^{N-1} \overline{\Phi_k(z_1)}\Phi_k(z_2) = \frac{\overline{\Phi_N^*(z_1)}\Phi_N^*(z_2) - \overline{\Phi_N(z_1)}\Phi_N(z_2)}{1 - z_2/z_1} \tag{7}$$

where $z_1 = e^{i\theta}$, $z_2 = e^{i\phi}$ and we have used the notation $\Phi_n^*(z) = z^n\Phi_n(1/z)$. We obtain the large-$N$ asymptotics of the polynomials by noting that the ratio

$$\frac{(q;q)_N(aq;q)_{N-k}}{(q;q)_{N-k}(aq;q)_N} = 1 + O(1/N).$$

Thus for $N \to \infty$

$$\Phi_N(z;q) \approx z^N \sum_{k=0}^{n} \frac{(a;q)_k(q^{1/2}/z)^k}{(q;q)_k} = z^N \frac{(aq^{1/2}/z;q)_\infty}{(q^{1/2}/z;q)_\infty} \qquad a^2q < 1 \tag{8}$$

where in the last line we have used the $q$-binomial theorem [12]. The two-level kernel in the large-$N$ limit can then be written in the general form

$$K_N \approx \frac{e^{i(N-1)(\theta-\phi)/2}}{2\pi} \left[\frac{(q^{1/2}z_1, q^{1/2}/z_2, aq^{1/2}/z_1, aq^{1/2}z_2;q)_\infty}{(q^{1/2}/z_1, q^{1/2}z_2, aq^{1/2}z_1, aq^{1/2}/z_2;q)_\infty}\right]^{1/2} \frac{\sin[N(\theta-\phi)/2 - \Delta]}{[\sin(\theta-\phi)/2]} \tag{9}$$

where the shift $\Delta$ is given by

$$\Delta = \mathrm{Im}\left[\ln \frac{(aq^{1/2}z_1, aq^{1/2}/z_2;q)_\infty}{(q^{1/2}z_1, q^{1/2}/z_2;q)_\infty}\right] \tag{10}$$

and we have used the notation $(x, y, \ldots, z; q)_n = (x;q)_n(y;q)_n \cdots (z;q)_n$. For fixed $q$, in the limit $\theta \approx \phi$, this can be simplified and we obtain

$$\Delta \approx 2(\frac{\theta-\phi}{2})\,\mathrm{Re}\left[e^{i(\theta+\phi)/2}\sum_{k=0}^{\infty}\frac{q^{k+1/2}}{1 - z_1q^{k+1/2}} - ae^{-i(\theta+\phi)/2}\sum_{k=0}^{\infty}\frac{q^{k+1/2}}{1 - aq^{k+1/2}/z_1}\right]. \tag{11}$$

Writing $1/(1 - xq^{k+1/2}) = \sum_{l=0}^{\infty}\left(xq^{k+1/2}\right)^l$ and summing over $k$ first, we obtain the following identity:

$$\sum_{k=0}^{\infty}\frac{q^{k+1/2}}{1 - xq^{k+1/2}} = \frac{q^{1/2}}{1-q}\sum_{l=0}^{\infty}\left(xq^{1/2}\right)^l \frac{1-q}{1-q^{l+1}}. \tag{12}$$

The factor $(1-q)/(1-q^{l+1}) \to 1$ for $q \ll 1$, while it is $1/(l+1)$ in the limit $q \to 1$. In both limits the sum can be explicitly evaluated; it turns out that the result for $q \to 1$ contains the $q \ll 1$ limit, giving a single expression valid for both limits. The result, in the limit $\theta \to \phi$, is

$$\Delta \approx \left(\frac{\theta-\phi}{2}\right)\frac{1}{1-q}\ln\left(\frac{1 - 2a\sqrt{q}\cos\theta + a^2q}{1 - 2\sqrt{q}\cos\theta + q}\right). \tag{13}$$

Equations (9) and (13) constitute the solution for large $N$ for the general model defined by (3), in the limit $\theta \approx \phi$.

We first consider the density of levels given by $\sigma(\theta) = K_N(\theta, \theta)$. Using equations (6), (9) and (13), we get

$$\sigma(\theta) \approx \frac{N}{2\pi}\left[1 + \frac{1}{(1-q)N}\ln\left(\frac{1 - 2a\sqrt{q}\cos\theta + a^2 q}{1 - 2\sqrt{q}\cos\theta + q}\right)\right]. \tag{14}$$

Note that the density has a finite-$N$ correction to the uniform density $N/2\pi$ of the circular ensemble. It is clear that in the $N \to \infty$ limit, the correction might survive only in the $q \to 1$ limit such that the product $(1-q)N$ is kept finite. This is precisely the special limit, namely $q = \mathrm{e}^{-1/N}$ and $a = q^{N/\lambda}$ that defines model (2), and as we argued in the beginning, this is indeed the limit where we expect the effect of localization to become observable in the spectral correlations. In the rest of our discussions we will restrict ourselves to this limit alone.

The expression (14) for the density of levels has one apparently very disturbing feature. Although it is properly normalized to $N$, the density actually becomes negative for sufficiently small values of $\theta$. In fact the condition for the density to remain positive for all values of $\theta$ is that the parameter $\lambda > \lambda_c = 2N(\sqrt{\mathrm{e}} - 1)$. For $1 \ll \lambda \ll \lambda_c$, the density is positive only for $\theta > \theta_c$ given by $2\sqrt{\mathrm{e} - 1}\sin(\theta_c/2) \sim 1/\lambda$. Thus with decreasing $\lambda$, i.e. increasing localization, $\theta_c$ increases. We will now show that the negative density for $\lambda < \lambda_c$ implies that there exists a gap in the spectrum for $\theta < \theta_c$.

In order to understand the density for $\lambda < \lambda_c$, we will briefly use an alternative approach based on the large-$N$ 'Coulomb-gas' approximation [13]. If we write $w(\theta) = \mathrm{e}^{-V(\theta)}$, we can interpret the right-hand side of (1) as $\mathrm{e}^{-H}$, where the effective 'Hamiltonian'

$$H = \alpha \sum_{m \neq n} \ln\left|2\sin\frac{\theta_m - \theta_n}{2}\right| - \sum_n V(\theta_n)$$

and the eigenvalues are given by the stationary condition

$$V'(\theta) = \alpha P \int_I \mathrm{d}\phi\, \sigma(\phi) \cot\frac{\theta - \phi}{2} \tag{15}$$

where $\sigma(\phi)$ is the density to be evaluated, $V'$ is the derivative of $V$ with respect to $\theta$, $P$ denotes a principal value integral, and the range $I$ of the integral is determined from the normalization $\int_I \mathrm{d}\phi\, \sigma(\phi) = N$. Expanding $\cot(A - B)$ and using the normalization, we get ($\alpha = 2$)

$$V'(\theta) = N\cot\frac{\theta}{2} + \csc^2\frac{\theta}{2} P\int_{\theta_c}^{2\pi-\theta_c} \mathrm{d}\phi\, \frac{\sigma(\phi)}{\cot\frac{\phi}{2} - \cot\frac{\theta}{2}} \tag{16}$$

where we have allowed for the possibility that the eigenvalues lie in the region $|\theta| \geqslant \theta_c$, $\theta_c \leqslant \pi$. For our model, $V(\theta) = -\ln(1 - \cos\theta)^{N/\lambda} + \text{constant}$. Using $x = [\cot(\theta/2)]/[\cot(\theta_c/2)]$ and $y = [\cot(\phi/2)]/[\cot(\theta_c/2)]$, we can rewrite (16) as

$$-N\left(1 + \frac{1}{\lambda}\right)\frac{bx}{1 + bx^2} = P\int_{-1}^{1} \mathrm{d}y\, \frac{f(y)}{x - y} \tag{17}$$

where we have defined $b = \cot^2(\theta_c/2)$, and $f(y)\mathrm{d}y = \sigma(\phi)\mathrm{d}\phi$. This integral can be inverted [14] to give

$$f(x) = -\frac{N(1+\lambda)b}{\lambda\pi^2}\sqrt{\frac{1-x}{1+x}}P\int_{-1}^{1}\sqrt{\frac{1+y}{1-y}}\frac{y}{1+by^2}\frac{\mathrm{d}y}{y-x}. \tag{18}$$

The integral can be evaluated explicitly, giving

$$\frac{\pi}{\sqrt{1+b}}\frac{1+x}{1+bx^2}.$$

Going back to the original variables, we obtain

$$\sigma(\theta) = \frac{N}{2\pi}\frac{1+\lambda}{\lambda}\sin\frac{\theta_c}{2}\sqrt{\cot^2\frac{\theta_c}{2} - \cot^2\frac{\theta}{2}} \qquad |\theta| > \theta_c. \tag{19}$$

The normalization condition gives

$$\sin\frac{\theta_c}{2} = \frac{1}{1+\lambda} \sim \frac{1}{\lambda} \qquad \lambda \gg 1.$$

This agrees with our previous result on the existence of the gap as well as its dependence on $\lambda$. A similar model, with $w(\theta) = e^{(2N/\lambda)\cos\theta}$ has been solved for the density in the saddle-point approximation in the context of the large-$N$ behaviour of $U(N)$ lattice gauge theories in two spacetime dimensions [15]. A similar gap was found (at $\theta = \pi$), which suggests that the result is not peculiar to the particular model we chose; in particular the results from our solvable model should be qualitatively valid for models involving qualitatively similar constraints on $\mathrm{Tr}(S + S^{\dagger})$.

The advantage of our solvable model is that we can go beyond the density and evaluate the two-level kernel from which all $n$-point correlation functions can be calculated. However, we cannot use equations (5) and (6) directly because of the gap in the spectrum. The existence of the gap suggests that we must allow for this possibility from the beginning, and replace (5) by

$$\frac{C}{2\pi}\int_{\theta_c}^{2\pi-\theta_c} \Phi_m(e^{i\theta}; q)\overline{\Phi_n(e^{i\theta}; q)}w(\theta)\, d\theta = \delta_{m,n}. \tag{20}$$

Although this means that the polynomials are no longer given exactly by (4), we note that for small $\theta_c$, the density in the large-$N$ limit is almost uniform everywhere except near the edges. If we restrict ourselves to this uniform density regime, far from the edges, then the only real effect of the gap is to affect the normalization. We have taken this into account simply by renormalizing the polynomials (4) by a factor $\sqrt{C}$ in (20) above. For small values of $\theta_c$, equivalent to large $\lambda$, the normalization constant is $C \approx 1/(1 + c/\lambda)$, where $c$ is a constant O(1). We will restrict our following discussions only to the regime $\theta \approx \pi$, where the density is approximately uniform, and the kernel $K_N(\theta, \phi)$ becomes translationally invariant:

$$|K_N(\theta - \phi)| \approx \frac{C}{2\pi}\left|\frac{\sin\left[\frac{1}{2}N(\theta-\phi)(1+1/\lambda)\right]}{\sin\left[\frac{1}{2}(\theta-\phi)\right]}\right| \tag{21}$$

where we have included the normalization constant $C$ explicitly, and $K(\phi - \theta)$ is the complex conjugate of $K(\theta - \phi)$. In order to compare with the random-matrix theories, we have to 'unfold' the spectrum by going to a new variable where the mean spacing between nearest levels is unity [1]. This is obtained by choosing the new variables $(\zeta, \eta) = (NC/2\pi)(1+1/\lambda)(\theta, \phi)$. In terms of these variables the two-level kernel becomes simply

$$|K(\zeta - \eta)| \approx C\left|\frac{\sin\left[\pi(\zeta-\eta)/C\right]}{\pi(\zeta-\eta)}\right|. \tag{22}$$

Note that this looks identical to the two-level kernel of the Gaussian random-matrix theory [1], if we define a new set of variables $(\zeta^*, \eta^*) = (1/C)(\zeta, \eta)$. However, in this new variable the mean spacing is not unity, but $1/C$, so the 'unfolding' of the spectrum will take us back to the variable $(\zeta, \eta)$.

The two-level kernel can now be used to calculate, e.g., the nearest-neighbour spacing distribution or the long-range spectral rigidity. To demonstrate the qualitative effects of localization, we will explicitly calculate the number variance for an interval $s$, defined as $(\delta n)^2 = \langle n^2 \rangle - \langle n \rangle^2$. Using $r = \zeta - \eta$, this is given in terms of the kernel as [1]

$$(\delta n)^2 = s - 2\int_0^s \mathrm{d}r\ (s-r)|K(r)|^2$$
$$= s[1-C] + \frac{C^2}{\pi^2}[\ln(2\pi s/C) + \gamma + 1] + \mathrm{O}(s^{-1}) \qquad C = \frac{1}{1+c/\lambda} \tag{23}$$

where $\gamma$ is Euler's constant. As $\lambda \to \infty$, the linear dependence on $s$ cancels exactly and the expression reduces to the universal logarithmic dependence on $s$ characteristic of the Wigner distribution. However, for any finite $\lambda$, there is a leftover linear dependence on $s$ with the slope increasing with decreasing $\lambda$ (increasing localization). This clearly signals a crossover from a Wigner towards a Poisson distribution (for which $(\delta n)^2 = s$) similar to that seen in the case of unbounded eigenvalues [6], and also similar to the crossover seen in numerical studies of the number variance for $S$-matrix eigenvalues [9] describing transport in mesoscopic conductors [7] as well as for $\Delta_3$ statistics (a related measure of the long-range spectral rigidity [1]) of the Flouquet-matrix eigenvalues describing time evolution of the Fermi-accelerator model [8]. Note that if $\lambda$ is related to a physical parameter like the conductance which itself scales with $N$, then starting from an intermediate case for finite $N$ as given in (23), the distribution will scale towards either Wigner or Poisson limit depending on whether $\lambda$ scales towards $\infty$ or 0 with increasing $N$.

We briefly point out that the general model (3) might include other physically interesting models. For example in the limit $a = 0$ and $q \to 1^-$, the function [17]

$$w(\theta) \to \exp\left[-\frac{1}{1-q}\cos^2\frac{\theta}{2}\right]$$

which is the model considered in [15].

In summary, we have constructed a one-parameter solvable model (as a special limit of a more general two-parameter solvable model) for the joint probability distribution of eigenvalues of unitary matrices which, in the large-$N$ limit, leads to a gap in the density. The gap increases as a function of the parameter. By analysing the effect of the gap on the number variance, we argued that the model qualitatively describes the effect of localization.

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